A simple proof of associativity and commutativity of LR-coefficients (or the hive ring)

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In this paper we propose a simple bijective proof of associativity and commutativity of Littlewood-Richardson coefficient or the hive ring ([13]).

1 Introduction

The ring of symmetric functions $\mathbb{Z}[x_1, \ldots, x_n]$ has a distinguished basis constituted of the Schur functions s_λ, where λ runs over the n-tuples of partitions, that is non-increasing tuples on non-negative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n))$. The coefficients $c^\lambda_{\mu,\nu}$ of the product

$$s_\mu s_\nu = \sum c^\lambda_{\mu,\nu} s_\lambda,$$

are known as the Littlewood-Richardson coefficients. These coefficients occur in many problems, see, for example, [10, 14].

From this definition obviously follow commutativity and associativity

$$c^\lambda_{\mu,\nu} = c^\lambda_{\nu,\mu}$$

$$\sum_\sigma c^\sigma_{\lambda,\mu} c^\pi_{\sigma,\nu} = \sum_\tau c^\tau_{\mu,\nu} c^\pi_{\lambda,\tau}. \tag{1}$$

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Littlewood and Richardson proposed combinatorial interpretation of the coefficient $c^\lambda_{\mu,\nu}$ as the cardinality of so-called LR-tableaux, skew tableaux of shape λ/μ and weight ν with the reverse lattice words reading (see, for example, [14]).

In [1] it was found a nice interpretation of LR-tableaux as special functions on a triangle grid, see also [2, 3, 4, 7, 12]. Due to this line of approach, LR-coefficients count integer-valued discretely concave functions on the triangle grid with the prescribed boundary values ([3, 4]) or hives ([2, 12]). In these terms the commutativity and associativity relations are non-trivial statements and give a hint for existing natural bijections among corresponding sets.

In [13] is proposed a proof of this relation in the language of hives, that is establishing a bijection between two sets of pairs of functions on the triangle grid. The second part of the proof in not transparent for us[1] and here we propose a simple proof of this bijection. In fact, we prove a bit stronger result of existence a polarized polymatroidal discretely concave function on the tetrahedron grid with prescribed boundary values on two adjoint faces of the tetrahedron (which, of course, has to be discretely concave functions on the corresponding triangular grids), for precise definitions see Section 3. As a consequence of this result we get the bijection (due to the polarization property, that is the *octahedron rule*). It is interesting to point out that we recognized the octahedron rule as a stronger version of the octahedron exchange axiom ([5]) or a part of the local exchange axiom ([8, 15]). The local exchange axiom allows one of equivalent ways to construct theory of polymatroidal discretely concave functions. Note that in dimensions $d \geq 3$ there is no needs for higher dimensional octahedron recursions to define polymatroidal discretely concave functions in $\mathbb{Z}^d$.

As a consequence of our main result, we also get a bijection

$$DC(\mu, \nu; \lambda) \cong DC(\nu, \mu; \lambda),$$

which correspond to the commutativity $c^\lambda_{\mu,\nu} = c^\lambda_{\nu,\mu}$.

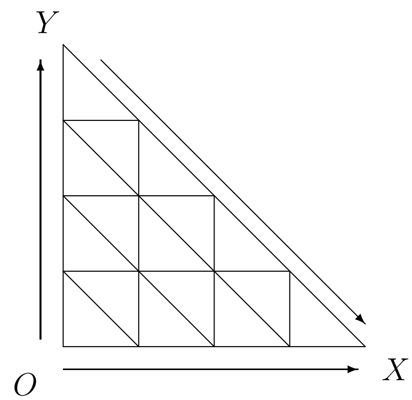
In the language of A-type crystals of Kashiwara, a similar associativity bijection can be obtained using twice the canonical commutativity isomorphism $R : A \otimes B \to B \otimes A$ ([6]) in the arrays category of crystals, namely applying $R : O_\mu \otimes (O_\nu \otimes O_\lambda) \to (O_\nu \otimes O_\lambda) \otimes O_\mu$ and then $R : O_\tau \otimes O_\mu \to O_\mu \times O_\tau$ for each irreducible component O_τ of $O_\nu \otimes O_\lambda$ we get an associativity bijection. Conjecturally, these bijections have to coincide (looks alike Conjecture 3 in [16]).

[1] Recently in [11] a sketch of another proof is proposed.

Acknowledgements. We thank Igor Pak for discussions and comments and directing us to the preprint [11].

2 Discretely concave functions on two-dimensional grid

We let $\Delta_n = \Delta_n(O, X, Y)$ to denote the two-dimensional grid of size n, that is the set of points $(i, j) \in \mathbb{Z}^2$ such that $i \geq 0$, $j \geq 0$ $\;i + j \leq n$. Here we depict the grid of size 4.



We consider a subclass of so called "discretely concave" functions defined on this grid Δ_n [3, 4]. Specifically, we cut the plane $\mathbb{R}^2$, which contains the grid Δ_n, by three types of lines $x = i$, $y = j$, $x + y = k$, where i, j and k run over the integers. These lines cut the triangle $\mathrm{co}(\Delta_n)$ into small (unitary) triangles as we depicted on the above Picture. Now, a given function f defined at the points of Δ_n we interpolate on each small triangle by affinity. As a result of this interpolation we get a continuous piece-wise linear function $\tilde{f}$ defined on the whole triangle $\mathrm{co}(\Delta_n)$.

Definition. A function $f : \Delta_n \to \mathbb{R}$ is *discretely concave* if the piece-wise linear interpolation $\tilde{f}$ is a concave function[2].

We can reformulate discrete concavity of a function f without using the

[2]Of course, we can define discretely concave functions on the whole lattice $f : \mathbb{Z}^2 \to \mathbb{R} \cup \{-\infty\}$, and due to the terminology in [5] such functions have to be called "$\mathbb{A}_2$-concave or two-dimensional polymatroidal concave function; in [15] such functions called $M^\natural$-concave; discretely concave functions on Δ_n called "hives" in [12]. We prefer to use this terminology from [4], since on the one hand it reflects the discreteness of the domain and on the another hand pointed out the crucial property of "concavity".

interpolation $\tilde{f}$. Namely we have to require validity of three types of "rhombus" inequalities

(i) $f(i,j) + f(i+1,j+1) \leq f(i+1,j) + f(i,j+1)$;

(ii) $f(i+1,j) + f(i+1,j+1) \geq f(i,j+1) + f(i+2,j)$;

(iii) $f(i,j+1) + f(i+1,j+1) \geq f(i,j+2) + f(i+1,j)$.

That is for each small rhombus inside the grid (or the whole $\mathbb{Z}^2$) the sum of values along the diagonal which belong to a cut line is greater or equal to the sum along the diagonal which does not belong.

Consider a function f on the grid Δ_n and consider its restriction to each side of the triangle: the base of the triangle, the left-hand side and the hypotenuse. Specifically, we have to orient these sides as depict on the previous picture and consider INCREMENTS of the function on each unit segment (arrow). Then, increments along the left-hand side constitute an n-tuple $\mu = (\mu_1, \ldots, \mu_n)$. It is easy follows from the rhombus inequalities of the type (i) and (iii) that $\mu_1 \geq \ldots \geq \mu_n$, that is μ is a partition. Analogously, we consider the increments along the hypotenuse and will get a non-increasing (the rhombus inequalities (ii) and (iii)) n-tuple $\nu = (\nu_1, \ldots, \nu_n)$, and along the base of the triangle we get non-increasing (the rhombus inequalities (i) and (ii)) n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$. This triple μ, ν and λ we call boundary *increments* of a function f. Obviously, the increments are invariant under adding a constant to f. Therefore, we have to consider functions modulo adding a constant or to require $f(0) = 0$.

The set of all discretely concave functions on Δ_n with increments (μ, ν, λ) we denote $DC_n(\mu, \nu, \lambda)$. This set is a polyhedron (possible empty) in the space of all functions on Δ_n. Obvious necessary conditions are non-increasing of μ, ν and λ and validity of the equality $\mu_1 + \ldots + \mu_n + \nu_1 + \ldots + \nu_n = \lambda_1 + \ldots + \lambda_n$. Of course for $n > 2$ these conditions are too far to be sufficient, for details see [10, 12, 4].

For integers' tuples (μ, ν, λ), of special interest are integer points of this polytope, that is integer-valued functions in $DC_n(\mu, \nu, \lambda)$. We let $DC_n(\mu, \nu, \lambda)(\mathbb{Z})$ to denote this set. Within the array model of A-type crystals [6], elements of this set encode the highest weight vectors in the tensor product of irreducible crystals $O_\mu \otimes O_\nu$, which span crystals isomorphic to O_λ. In particular, from that follows that cardinality of this set coincides with the Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$, i.e.

$$|DC_n(\mu, \nu, \lambda)(\mathbb{Z})| = c^\lambda_{\mu,\nu}. \tag{2}$$

The first proof of (2) is, implicitly, in [1], for other proofs see Appendix in[2], [3, 7, 12].

Thus, in the language of discretely concave functions, the associativity

formula

$$\sum_\sigma c^\sigma_{\lambda,\mu} c^\pi_{\sigma,\nu} = \sum_\tau c^\tau_{\mu,\nu} c^\pi_{\lambda,\tau}$$

reads as coincidence of cardinalities of the following two sets

$$\coprod_\lambda DC_n(\mu,\nu,\lambda)(\mathbb{Z}) \times DC_n(\lambda,\pi,\sigma)(\mathbb{Z})$$

and

$$\coprod_\tau DC_n(\nu,\pi,\tau)(\mathbb{Z}) \times DC_n(\mu,\tau,\sigma)(\mathbb{Z}).$$

Of course, this gives a hint that it should exist a "natural" bijection between these sets. In [13] is proposed a construction (we will call it KTW-construction) of a bijection between sets

$$\coprod_\lambda DC_n(\mu,\nu,\lambda) \times DC_n(\lambda,\pi,\sigma)$$

and

$$\coprod_\tau DC_n(\nu,\pi,\tau) \times DC_n(\mu,\tau,\sigma)$$

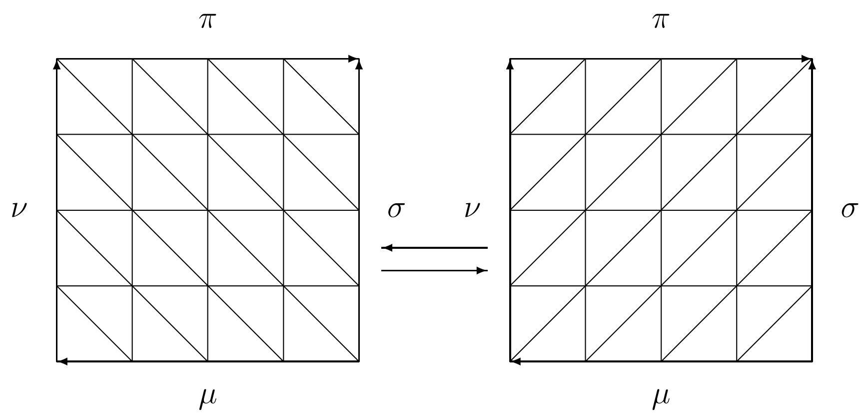
(here μ, ν, λ etc. are real-valued, not necessary integer-valued), which is, first, piece-wise linear and, second, sends integer points into integer points[3]. As a consequence of the main Theorem in the next section we will get a proof of that the KTW-construction is a bijection indeed.

3 Functions on three dimensional grid

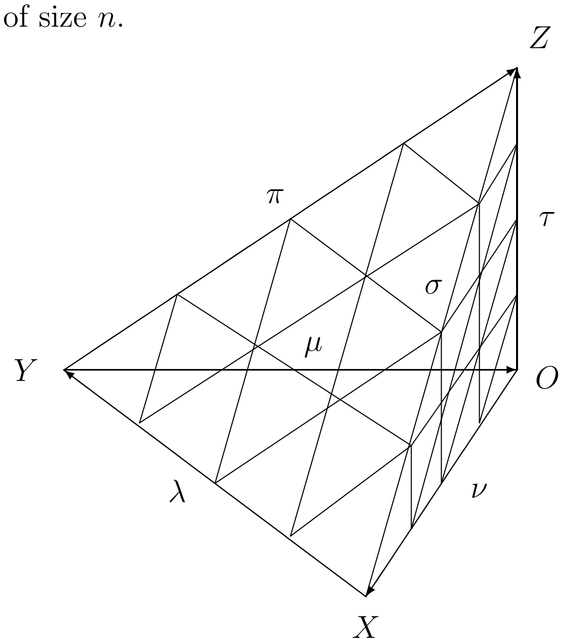
Of course we have to recall nice and transparent KTW-construction. For that we slightly transform the problem. Let we have two functions f and f' on two copies of the grid Δ_n such that λ, the tuple of increments of f along the hypotenuse, is also the tuple of increments of f' along the base of the triangle. Then we can glue these copies along the sides on which f and f' have equal increments. As a result we get a function on the "square" grid, which satisfies the rhombus inequalities for almost all rhombus in this square grid, namely, for all rhombus except those which have diagonals located on the north-west south-east diagonal of square $\{0 \leq x, y \leq n\}$. Now, given this function we have to construct a function on the same square with the

[3]The final step of the proof in [13] is not transparent and it is not clear to us how Theorem 1.6 from [9] can be used to finish the proof.

same boundary increments, but which is piece-wise linear with respect to the mirror triangulation, see the Picture.



Let us consider these 4 triangles as four faces of the standard tetrahedron $\Delta_n(O,X,Y,Z)$ of size n.



Initial functions f and f' are given on the *ground* $\Delta_n(O,X,Y)$ and on the *ceiling* $\Delta_n(X,Y,Z)$, respectively. The outcome functions are defined on the *walls* $\Delta_n(O,Y,Z)$ and $\Delta_n(O,X,Z)$. (On the picture we triangulated the ground and one of the walls, analogously have to be triangulated the ceiling and another wall.) The KTW-construction propagates a function from the ground and ceiling to all integer points of the tetrahedron

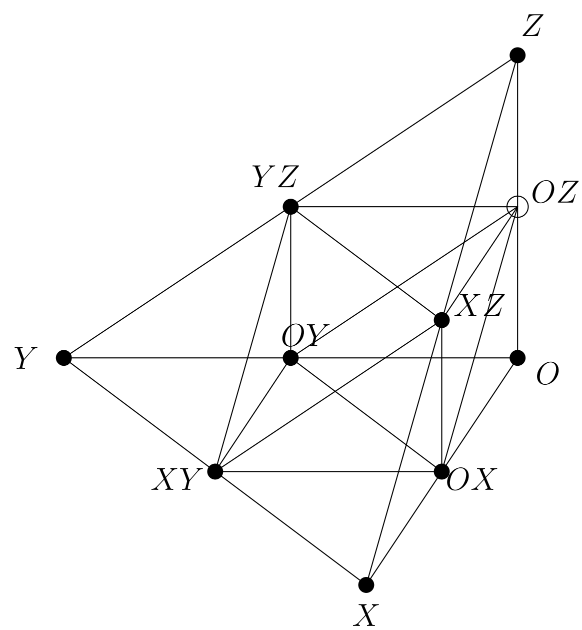
$$\Delta_n(O,X,Y,Z) = \{(x,y,z) \in \mathbb{Z}^3, x,y,z \geq 0, x+y+z \leq n\}.$$

This propagation procedure runs through the so-called *octahedron recursion*. It is clear from this recursion that it is defined by piece-wise linear functions and is invertible. But that is not clear that starting from discretely concave functions f and f' on the ground and ceiling we will end up with discretely concave functions on the walls of our "house". As we pointed out, the non-trivial step of the proof in [13] is made by addressing to the "tropical Laurent polynomials" statement in [9], and how this works is still unclear to us.

We are going to identify the class of functions on the three-dimensional grid, which are obtained due to this recursion, and to prove that they form a subclass of the class of three-dimensional discretely concave functions on the grid $\Delta_n(O, X, Y, Z)$. Discrete concavity of such functions on the faces of the tetrahedron holds true due to the definition.

Let us consider functions defined on the three-dimensional grid $\Delta_n(O, X, Y, Z)$. (The convex hull of this grid we let to denote by the same symbol and let to call it the *tetrahedron*.)

We cut $\mathbb{R}^3$ (and the tetrahedron) by four series of planes $x = i$, $y = j$, $z = k$, $x + y + z = l$, where i, j, k and l run over integers (from 0 until n for the tetrahedron). As a result the tetrahedron is cut into small "unitary" pieces: simplexes and octahedrons. (The matter has to be clear from the following picture with $n = 2$.)



The main attention we pay to the octahedrons. All octahedrons are "similar" and one is obtained from another by integer translations. Each

octahedron has three main diagonal, or equivalently, three pairs of antipodal vertices: OX and YZ, OY and XZ, OZ and XY. The latter pair we distinguish among others and the corresponding diagonal (parallel to the vector $(1,1,-1)$) in each octahedron we call the *main diagonal*.

Reasons why we distinguish this diagonal might be seen in the simplest case $n=2$. In this case, there is the sole point OZ which does not belong to the ground face and ceiling face, and we have to propagate a function to this point. The rule for the propagation, called the *octahedron rule* in [13], prescribes to set

$$f(OZ) = \max(f(OX)+f(YZ), f(OY)+f(XZ)) - f(XY).$$

In words this rule (the octahedron recursion) says: we have to set the value at the "free" end of the main diagonal such that the sum of values at the vertices of this main diagonal has to equal the maximum over the sums with respect to other (non-main) diagonals.

This leads us to the following

Definition. A function $f:\mathbb{Z}^3 \to \mathbb{R}\cup\{-\infty\})$ (or $f:\Delta_n(O,X,Y,Z)\to\mathbb{R}$) is said to be *polarized*, if, for any unitary octahedron, the sum of values of f at the vertexes of the main diagonal is equal to the maximum of the sums of f on vertexes on the other diagonals.

It is obvious, that for any values at the points of the ground and ceiling faces of the grid $\Delta_n(O,X,Y,Z)$, there exists a unique polarized function on $\Delta_n(O,X,Y,Z)$ with these boundary values. This is the essence of the octahedron recursion.

We are going to prove that discrete concave "initial" data at the ground face OZY and the ceiling face XYZ produce a polarized function which possesses the three-dimensional discrete concavity. Without going deeply in details of this concept (which has sense in any dimension, see [5, 15]), we give definitions which are of use here.

Recall that discrete concavity in dimension 2 is related to the rhombus inequalities. In dimension 3, we have to require inequalities for the octahedrons and the rhombus inequalities for each of 4 types cutting planes. Namely, in each such a cutting plane we have three types of rhombus and the rhombus inequalities require the sum at the vertices on the rhombus diagonal lying on a cutting plane exceed the values at the another diagonal. The octahedrons' inequalities says that values at two diagonal have to be equal to the maximum over three values at the pairs of antipodal vertexes. Functions with these properties are called polymatroidal discretely concave functions in [5] and $M^\natural$-concave in [15]. The octahedron rule is a stronger requirement and implies validity of the octahedron inequalities.

Definition. A function f on the three-dimensional grid $\Delta_n(O, X, Y, Z)$ is a *polarized discretely concave function* if f is polarized and all rhombus' inequalities hold true. We let $PCPM_n$ to denote this set of functions.

It is clear that restrictions of a $PCPM_n$-function f to any cutting plane, and, hence, to the ground face, or to the ceiling face, or to the walls, are two-dimensional discretely concave functions. Thus, we will obtain that the KTW-construction is a bijection if we get the following

Theorem. *Let f be a polarized function on the grid $\Delta_n(O, X, Y, Z)$ and let the restrictions of f to the ground face and the ceiling face be two-dimensional discretely concave functions. Then $f \in PCPM_n$.*

4 Proof of Theorem

In the beginning we consider the case $n = 2$ (see the picture above). Let f be a polarized function on $\Delta_2(O, X, Y, Z)$. We have to check the rhombus inequalities. In this case all rhombus are located on faces of the tetrahedron. For rhombus located on the ground or ceiling faces, the corresponding inequities hold true due to the assumptions. Thus, we have to consider rhombus on the walls. Since the walls are symmetrical, it suffices to verify the rhombus inequalities for the rhombus on the wall OXZ. This wall contains three rhombus, the rhombus O, OZ, XZ, OX, the rhombus X, OX, OZ, XZ and the rhombus Z, OZ, OX, XZ. Since the latter two rhombus are similar, we have to verify the inequalities for the first and the second.

The first rhombus. We have to check that

$$f(OZ) + f(OX) \geq f(XZ) + f(O).$$

Since the vertexes OZ and XY form the main diagonal in the octahedron, from the polarization condition, we have the inequality

$$f(OZ) + f(XY) \geq f(XZ) + f(OY).$$

From the rhombus inequality, for the rhombus O, OX, XY, OY on the ground face, we have

$$f(OX) + f(OY) \geq f(O) + f(XY).$$

Summing up these two, we get the required inequality.

The second rhombus. We have to show

$$f(OX) + f(XZ) \geq f(X) + f(OZ).$$

From the polarization we have either the equality

$$f(OZ) + f(XY) = f(XZ) + f(OY),$$

or

$$f(OZ) + f(XY) = f(OX) + f(YZ).$$

Assume the first equality holds true. Then, from the inequality for the rhombus X, XY, OY, OX lying on the ground face, we get the inequality

$$f(OY) + f(X) \leq f(XY) + f(OX).$$

Summing up it with the first equality, we get the required inequality.

Assume the second equality holds true. Then using the rhombus X, XY, YZ, XZ from the ceiling face, we have the rhombus inequality

$$f(X) + f(YZ) \leq f(XY) + f(XZ).$$

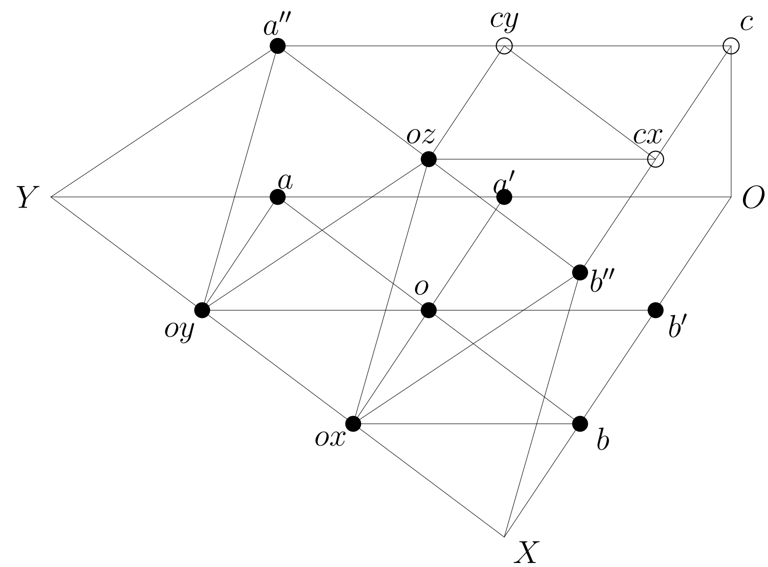
Summing up this inequality and the second equality, we get the required inequality.

Thus, the case $n = 2$ is proven.

The general case. For this case, we need the following

Lemma. *Let a polarized function f be such that its restrictions to the ground and ceiling faces of $\Delta_n(O, X, Y, Z)$ are discretely concave functions. Then its restriction to any cutting plane parallel to the ground or ceiling face, i.e. sets of the form either $\{z = a\} \cap \Delta_n(O, X, Y, Z)$ or $\{x + y + z = d\} \cap \Delta_n(O, X, Y, Z)$, a, $c \in \mathbb{Z}$, is a two-dimensional discrete concave fucntion.*

Proof. We begin by checking the rhombus inequalities for rhombus located in the plane $z = 1$, and which have non-empty intersection with the ceiling face. This case might be handled by the case $n = 3$, see the next picture.



On the first floor ($z = 1$) there is two essentially different rhombus: c, cx, oz, cy and a'', oz, cz, cy. We have to check two corresponding inequalities:

$$f(cy) + f(cx) \geq f(c) + f(oz)$$

and

$$f(cy) + f(oz) \geq f(a'') + f(cx).$$

The first inequality: Since c is one of the vertexes of the main diagonal $\{c, o\}$, the sum $f(c)+f(o)$ is equal to either $f(cy)+f(b')$ or $f(cx)+f(a')$. Suppose $f(c) + f(o) = f(cx) + f(a')$ holds true. Since cy is the vertex of the main diagonal $\{cy, oy\}$, we have $f(cy) + f(oy) \geq f(a) + f(oz)$. Finally, for the ground rhombus o, oy, a, a', we have the rhombus inequality $f(a) + f(o) \geq f(a')+f(oy)$. Summing up these inequalities and the equality $f(cx)+f(a') = f(c)+f(o)$, we get the desired inequality. The case $f(c)+f(o) = f(cy)+f(b')$ is handled analogously.

The second inequality: Since the points cy and oy constitute the main diagonal, we have, from the polarization, $f(cy)+f(oy) \geq f(a'')+f(o)$. Now, for the main diagonal $\{cx, ox\}$, we have at least one of two equalities either $f(cx) + f(ox) =$ is equal to $f(b) + f(oz)$, or $f(cx) + f(ox) = f(o) + f(b'')$.

Suppose the first equality $f(b) + f(oz) = f(cx) + f(ox)$ holds true. From the ground rhombus $\{ox, oy, o, b\}$ we have $f(o) + f(ox) \geq f(oy) + f(b)$. Summing up $f(cy)+f(oy) \geq f(a'')+f(o)$, $f(b)+f(oz) = f(cx)+f(ox)$ and $f(o) + f(ox) \geq f(oy) + f(b)$, we get the required inequality.

Suppose the second equality $f(o) + f(b'') = f(cx) + f(ox)$ holds true. In this case, we use a rhombus inequality for rhombus $\{ox, ox, oz, b''\}$ located on the CEILING face. Namely, summing up the inequality $f(ox) + f(oz) \geq f(oy) + f(b'')$ and the above equality we get the required second inequality.

Now, one can see that rhombus inequality for the rhombuses being translations of the rhombus $\{oz, cy, c, cx\}$ by integer vectors, follow from the polarization and rhombus inequalities on the ground face. Thus, we get these inequalities. For a rhombus being a translation of the rhombus $\{b'', oz, cy, cx\}$ or $\{a'', cy, cx, oz\}$ we need rhombus inequalities for rhombuses which have an edge on the ground face. Thus, this step allows us to get rhombus inequalities for such rhombus which have a common edge either on the ceiling face or the ground face. Thus step by step, we obtain the rhombus inequalities for all rhombuses on the planes parallel either the ground face or the ceiling face.

Now all is prepared to prove the theorem. It remains to verify rhombus inequalities for rhombuses located on planes parallel to the walls. Pick a rhombus, say located on a plane parallel to the wall OXZ. Then take the integer translation of the tetrahedron Δ_2 of size 2, which contains this rhombus. By Lemma, the function f is discretely concave on the ceiling and ground faces of this tetrahedron. Since f is polarizes, from the proof for the case $n = 2$ follows that f is a polarized discrete concave on this tetrahedron and hence the desired rhombus inequality holds true. Q.E.D.

5 Commutativity

Here we apply the main theorem to establish a bijective proof of the commutativity

$$c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}.$$

Namely we obtain an isomorphism

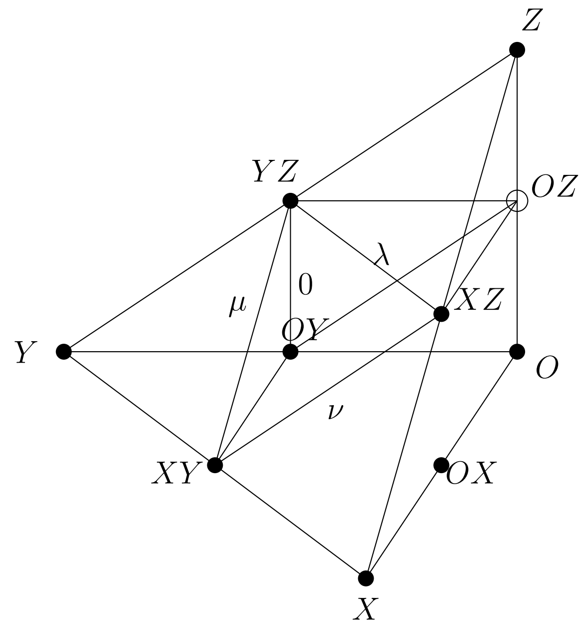
$$DC(\mu, \nu; \lambda) \cong DC(\nu, \mu; \lambda). \tag{3}$$

Such kinds of isomorphisms in [16] are called *fundamental symmetries*. In [6], this kind of symmetry was obtained using the natural isomorphism $R : A \otimes B \to B \otimes A$ in the array category of crystals. In [11] such kind of a bijection is called commutor.

So, let $f \in DC(\mu, \nu; \lambda)$ be a discretely concave function on the triangle grid Δ_n. Consider the top half of the octahedron inscribed in the tetrahedron $\Delta_{2n}(O, X, Y, Z)$. This half-octahedron is a polymatroid and there exist

polymatroidal discretely concave functions on this set[4]. Let us call a polymatroidal discretely concave function f defined on a polymatroidal set[5] $P \subset \mathbb{Z}^3$ a *polarized* if the octahedron rule holds true for each octahedron which has all vertices in P.

Now, we are looking for a polarized discretely concave function on this half of the octahedron, which is equal to f on the face of this half of the octahedron which is located on the ceiling of $\Delta_{2n}(O, X, Y, Z)$ and equals the separable function $p_\mu \in DC(0, \mu; \mu)$ on the face $\{XY, YZ, OY\}$ (see the next Picture).



Assume such a function F exists, then since F is a polymatroidal discretely concave function, for the unitary octahedrons which are cut by the face $\{XY, XY, OZ, OY\}$, the value of F at the vertex which does not belong to the octahedron $co\{XY, XY, OZ, OY, YZ\}$ equals $-\infty$. Thus for a rhombus on the face $\{XY, XY, OZ, OY\}$ which obtained after cutting this face by planes $z = a$, $x + y + z = b$, a and $b \in \mathbb{Z}_+$, we have the rhombus equality $F(A) + F(C) = F(B) + F(D)$ for the pair of antipodal vertices $\{A, C\}$ and $\{B, D\}$ of the rhombus. From this follows that increments of F along the edge $\{XZ, OZ\}$ is μ and along the edge $\{OY, OZ\}$ is ν. Since F is the

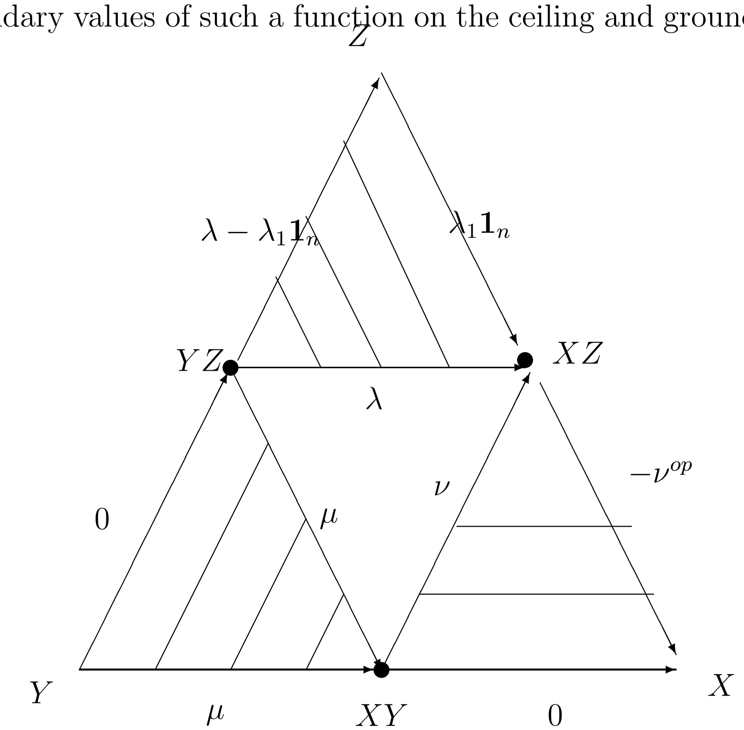
[4]Recall that polymatroidal discretely concave functions have efficiency domains of the form of polymatroids (integer points), and a polymatroidal DC function being restricted to a polymatroid, which is a subset of the efficiency domain, remains polymatroidal DC function. We understand restriction as setting $-\infty$ outside the restriction set.

[5]A subset $P \subset \mathbb{Z}^3$ is a polymatroidal set if $co(P)$ is a polymatroid, i.e. each edge of $co(P)$ is parallel to some vector of the set $\{e_i, e_i - e_j\}$, i, $j = 1, 2, 3$, and $P = co(P) \cap \mathbb{Z}^3$.

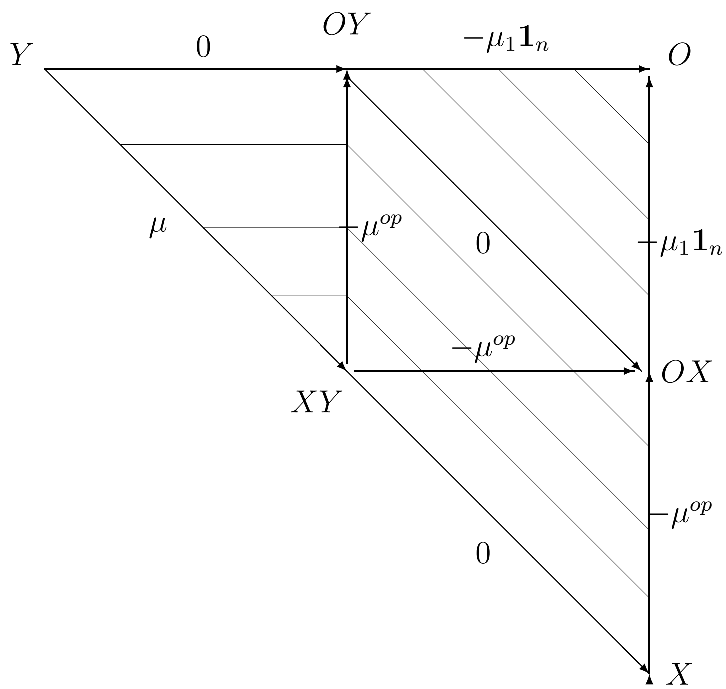
zero constant on the edge $\{YZ, OY\}$, from discrete concavity on the face $\{OY, YZ, OZ\}$ follows that the increments of F along the edge $\{YZ, OZ\}$ is ν. Thus F at the face $\{YZ, OZ, XZ\}$ is indeed a function from $DC(\nu, \mu; \lambda)$. Since F is polarized we get a desired bijection.

Proposition. *There exists a polarized discretely concave function on the half of the octahedron* $co\{XY, XY, OZ, OY, YZ\}$ *such that its restriction to the face* $\Delta_n(XZ, XY, XZ)$ *is equal to a given function of* $DC(\mu, \nu; \lambda)$ *and its restriction to* $\Delta_n(OY, YZ, XY)$ *is equal to the unique function of* $DC(0, \mu; \mu)$.

Proof. We proceed to find an appropriate polarized discretely concave function on the whole simplex $\Delta_{2n}(O, X, Y, Z)$. On the next two pictures we draw the boundary values of such a function on the ceiling and ground faces.



On this Picture we draw an extension of a discretely concave function defined on the grid $\Delta_n(YZ, XZ, XY)$ to the ceiling $\Delta_{2n}(X, Y, Z)$. We draw lines outside the triangle $co(YZ, XZ, XY)$ of constancy values of the extended function (ν^{op} denotes the non-decreasing tuple $(\nu_n, \dots, \nu_1)$ and $\mathbf{1}_n := (1, \dots, 1)$).



On this Picture we draw the function on the ground face, the boundary increments and the constancy levels determine a discretely concave function on the ground face.

Thus, due to the main theorem, there exists a polarized discretely concave function f on $\Delta_{2n}(O, X, Y, Z)$, which has restrictions to the ceiling and ground faces depicted on the above two pictures.

Let us prove that this function f restricted to the rhombus $co\{X, XZ, OZ, OX\}$ (on the wall $\Delta_n(O, X, Z)$) is a separable functions, i.e. for each unitary rhombus in this rhombus and being homothetic to it, we have the rhombus equality. (Note that these rhombuses are not located on the cutting planes.)

Assume this is done, then the tuple of increments along the side $\{OY, OZ\}$ is ν, hence that tuple along $\{YZ, OZ\}$ is also ν. Thus on the grid $\Delta_n(OZ, XZ, YZ)$ we get a function from $DC(\nu, \mu; \lambda)$ and so the desired polarized function on the half of the octahedron is obtained.

Thus, we have to prove validity of the rhombus equalities in the rhombus $co\{X, XZ, OZ, OX\}$.

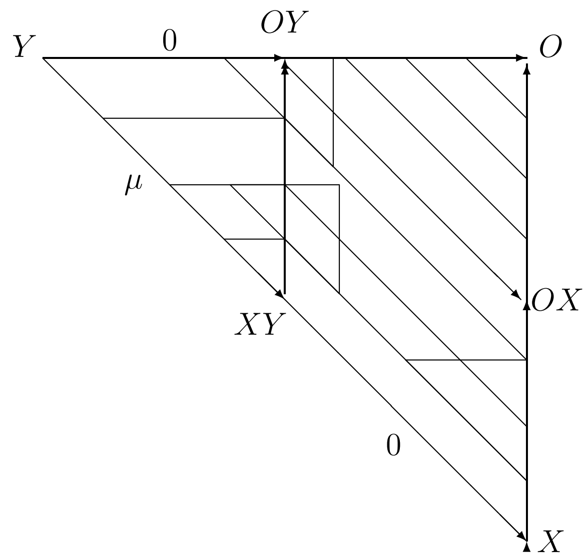
For this we use the following property of polarized functions on $\Delta_2(O, X, Y, Z)$:

Let $h \in PCPM_2$ and let $h(XY) + h(OX) = h(X) + h(OY)$. Then

$f(XZ) + f(OX) = f(X) + f(OZ)$ *holds true.*

In fact, since there holds $h(XZ) + h(OX) \geq h(X) + h(OZ)$ and due to $h(XY) + fh(OX) = h(X) + h(OY)$, we get $h(XY) + h(OZ) \leq h(OY) + h(XZ)$. Since $f \in PCPM_2$, the latter inequality holds as the reverse inequality and so is the equality indeed, that is $f(XZ) + f(OX) = f(X) + f(OZ)$ as claimed.

Now, having covered the trapezoid $co\{OY, O, X, XY\}$ by the ground triangles of the all integer translations of $\Delta_2(O, X, Y, Z)$, which intersect the interior of the trapezoid (see the next Picture, where we draw some of screening triangles), and applying the above claim to the corresponding tetrahedrons, we obtain the same shape of the constancy levels of f on $\{z = 1\} \cap co\{OY, O, X, XY, XZ, OZ\}$ as on the trapezoid $co\{OY, O, X, XY\}$. Lifting higher and higher we get that f satisfies all desired rhombus equalities. Thus, this completes the proof of Proposition.



Remark. It is not difficult to show that the bijection constructed above coincides with the bijection in [11].

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